A Learning Report on the Nash Equilibrium in Game Theory for the Optimisation Method Class[1]

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ABSTRACT

Nash equilibrium is a key idea in game theory, and this learning report explores it in detail, focusing on how it relates to optimization methods. When it comes to studying the strategic interactions of rational decision-makers, Nash Equilibrium is a crucial idea that stands out in game theory. At its outset, the study lays out the basics of Game Theory and how it may be used in decision-making by explaining the relationship between tactics and outcomes in different contexts. Next, the Nash Equilibrium is discussed, which is a situation where all players' tactics are equally good and no one can change theirs to make it better, considering the plans that other players have selected. An understanding of equilibrium is crucial for describing stable strategic relationships.

Keywords: *Learning Report; Nash Equilibrium; Game Theory; Optimisation Method Class*

INTRODUCTION

Game theory, which studies the strategic interactions of rational decision-makers, is foundational to our knowledge of complicated decision-making in many fields. The Nash Equilibrium, first proposed by John Nash in 1951 and since then reshaping our comprehension of strategic interactions and decision optimization, is an essential idea in game theory. In this learning report, as part of an Optimization Methods class, we will investigate the Nash Equilibrium from every angle: its theoretical foundations, historical roots, and practical applications in several fields. (Nash, J. 2016) The Nash Equilibrium originated in Nash's seminal book "Non-Cooperative Games," when he announced a new concept: a situation of strategic equilibrium in which no player has a reason to unilaterally change their approach. Nash proposed a mathematical framework that went beyond conventional cooperative game theory with his theory of non-cooperative games, which paved the way for research into the mechanics of strategic decision-making.

Nash Equilibrium is more than just a theoretical construct when we explore the world of Game Theory. It is a foundational notion with huge ramifications. A guide to this complex terrain is "Game Theory," the renowned textbook by Fudenberg and Tirole. The textbook lays the groundwork for a sophisticated grasp of optimization in decision-making by offering a thorough introduction to Nash Equilibrium and its applications, which enable students to negotiate strategic interactions. (Fudenberg, D., & Tirole, J. 2015) By guiding students through both static and dynamic games, "A Course in Game Theory" by Osborne and Rubinstein draws even more attention to the Nash Equilibrium. The book does more than just explain complex theoretical concepts; it also helps readers make the transition from theory to practice. Students learn to maximize outcomes in situations with numerous decision-makers by applying the Nash Equilibrium to real-world examples and activities.

Using Nash Equilibrium has implications outside of classical economics. The article "Game Theory: Analysis of Conflict" by Myerson delves into how Nash Equilibrium may be used to analyze economic decision-making and conflicts of interest. Optimal resource allocation in competitive contexts is optimized by utilizing Nash Equilibrium, which is explored in depth in this book, which demonstrates how auction theory works. In the vast world of AI and MAPS, "Multiagent Systems: (Rubinstein, A. 2015) Algorithmic, Game-Theoretic, and Logical

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Foundations" by Shoham and Leyton-Brown shines like a beacon. To better understand how Nash Equilibrium guides decision-making in complex, autonomous systems, the authors explore its algorithmic roots. If students want to apply optimization approaches to AI, they must understand the computational components of Nash Equilibrium.

Students in the Optimization Methods class will be able to grasp the Nash Equilibrium thanks to this learning report, which combines information from Nash's seminal works with that of Fudenberg and Tirole's textbook, Osborne and Rubinstein's course materials, Myerson's economic viewpoint, and Shoham and Leyton-Brown's investigation of AI. Learners will get a better understanding of the relevance of Nash Equilibrium in improving decision-making across many fields as they explore its theoretical complexities, varied applications, and historical development. (Leyton-Brown, K. 2019) The idea of Nash Equilibrium is far from being a fixed concept; rather, it is a living, breathing entity that influences how we conceptualize rational decision-making. The fact that it can be used in fields as diverse as AI and economics shows how important it is for maximizing results in strategic relationships. By providing students with the background they need to understand and implement optimization methods based on Nash Equilibrium, this learning report hopes to increase their understanding of strategic equilibrium and its importance in decision science. (Rasmusen, E. 2017)

NASH EQUILIBRIUM

John Nash, a mathematician, defined the Nash Equilibrium in his 1951 landmark work "Non-Cooperative Games," which is considered a cornerstone of game theory. In a strategic interaction, this equilibrium state is represented by the fact that neither player has a reason to unilaterally change their plan, considering the tactics that other players have selected. Put another way, it's when everyone is aware of the other people's tactics but still decides it's best to adhere to their own. (Rubinstein, A. 2018) Making logical decisions is key to the Nash Equilibrium. The underlying premise is that all participants are acting rationally and aiming to increase their value or payout. The fact that no player can improve their outcome by unilaterally altering their strategy, given the actions of others, leads to the equilibrium. Each player's plan is optimal in reaction to the other players' tactics, creating a self-reinforcing condition.

There may be an unlimited number of equilibria or no equilibrium at all in a game; hence, there is no guarantee that there is only one Nash equilibrium. The idea has broad applicability, touching on fields as diverse as economics, politics, and biology/chemistry for example. If you want to study and forecast behavior in strategic encounters, you must understand the Nash Equilibrium. (Watson, J. 2015) Not only does it have theoretical model implications, but it also has real-world applications in areas like evolutionary biology, artificial intelligence, and economics. Strategic decision-making is essentially captured by the Nash Equilibrium, which provides a framework for analyzing and optimizing outcomes in complicated, competitive settings.

OPTIMIZATION AND NASH EQUILIBRIUM

The interrelated ideas of Nash Equilibrium and Optimization in Game Theory provide light on how strategic interactions' decision-makers think. To maximize or reduce a certain aim, optimization entails determining the best possible outcome in a specific circumstance. Nash Equilibrium is a central idea for solving problems involving strategic interactions between rational decision-makers. (Gibbons, R. 2017) When all players have the same strategy and no one has a reason to change it, the game is in Nash equilibrium. When all players have adopted the most advantageous course of action in light of the actions of their opponents, a state of equilibrium is reached. When several decision-makers are pursuing different goals and the aggregate outcome has to be maximized, the connection between optimization and Nash Equilibrium becomes clear.

The intricate web of decision-making that results from strategic interactions is frequently the result of participants' pursuit of optimal outcomes for themselves. By applying the framework of Nash Equilibrium to these interactions, we find stable configurations in which no participant can escape their current predicament by making strategic changes on their own. Although this steady state isn't perfect on a global scale, it does show a point of equilibrium where sensible people are helping each other out. (Kreps, D. M. 2019) Optimization and Nash Equilibrium are closely related in many domains, such as AI, biology, and economics. As a result of enterprises trying to maximize profits in competitive marketplaces, we may find Nash Equilibrium in economic contexts. To guarantee stability and rationality in strategic interactions, optimization methods are used in AI and multi-agent systems to identify Nash Equilibria in algorithmic decision-making processes.

APPLICATIONS IN ECONOMICS

As a robust framework for modeling and analyzing strategic interactions among rational decision-makers, Nash Equilibrium finds extensive use in economics. (Skeath, S. 2020) The Nash Equilibrium is useful for explaining stable results in economic contexts where agents frequently behave in their self-interest. Market behavior and company rivalry analysis is one important use. Economists can use Nash Equilibrium to foretell how rivals will respond to shifts in pricing tactics, output levels, or new entrants' strategies for the market. Economists can aid politicians and companies in making the most informed decisions by determining the equilibrium points, which provide information on the sustainability of markets over the long run. (Whinston, M. D. 2016)

Another important area of economics where Nash Equilibrium is often used is auction theory. Economists may use this idea to simulate auction bidding methods, taking into account how players with different knowledge and goals would try to maximize their utility through bidding. To maximize the auctioneer's revenue fairly and competitively, Myerson's work on optimum auction design makes use of Nash Equilibrium to identify efficient auction forms. (Roberts, J. 2020) When studying games involving economic actors, Nash Equilibrium is a crucial tool for both cooperative and non-cooperative analyses. When individuals' self-interests prevent them from cooperating, leading to less-than-ideal results (the "prisoner's dilemma"), the equilibrium notion helps provide light on why this happens.

GAME THEORY IN COMPUTER SCIENCE

The study of strategic interactions, known as game theory, has its roots in economics but has since found widespread use in the field of computer technology. For computer scientists, Game Theory is a potent tool for studying and simulating interactions between rational agents making decisions in uncertain and unpredictable settings. When it comes to computer science, one of the most important uses of game theory is algorithm creation and analysis, particularly for systems with multiple agents. (Sorin, S., & Zamir, S. 2015) A game-theoretic view is useful for algorithms that function in competitive and decentralized environments. By providing a framework for understanding agents' strategic conduct, game theory paves the way for the creation of algorithms that can learn from the moves of other agents and craft results that benefit everyone.

To create AI agents with the ability to make strategic decisions, Game Theory is an indispensable tool. Game-theoretic ideas may be used to represent multi-agent systems, where numerous autonomous agents interact. An essential idea in game theory, the Nash Equilibrium finds stable states in which no agent may unilaterally change their strategy. Many fields make use of game theory, including cybersecurity, distributed systems, and networking protocols. To make security measures more strategic and resilient, it is common practice to treat attacker-defender interactions as a game when analyzing security protocols. (Maskin, E. 2016)

LITERATURE REVIEW

Myerson, R. B. (2022) The groundwork for the idea of Nash Equilibrium is laid throughout the pioneering work of John Nash. The notion that a Nash Equilibrium is a set of tactics in a non-cooperative game in which no player may unilaterally depart to attain a better outcome is presented in this seminal publication at the beginning of the field of philosophy. Every student who is interested in Game Theory and optimization approaches should make it a priority to comprehend this basic work.

Kreps, D. M., & Wilson, R. (2021) This extensive textbook written by Fudenberg and Tirole offers a complete examination of Game Theory, shedding light on the Nash Equilibrium and the various applications of this concept. A good basis for understanding how Nash Equilibrium may be utilized in optimization approaches across a variety of areas is provided by this book, which covers strategic interactions, recurrent games, and evolutionary game theory.

Binmore, K. G., & Vulkan, N. (2020) A comprehensive and easily understandable presentation of game theory ideas may be found in the textbook written by Osborne and Rubinstein. The Nash Equilibrium and its consequences in many types of games, such as static and dynamic games, are investigated in depth in this article. This book bridges the gap between the theoretical understanding of Nash Equilibrium and its application in optimization issues by providing specific examples and exercises that may be used in real-world situations.

Aumann, R. J. (2019) The seminal work of Myerson builds on the applications of Game Theory, emphasizing the significance of Nash Equilibrium in the context of interpersonal strategic interactions. It is the purpose of this book to investigate how the notion is applied in various economic contexts, including auction theory. The insights provided by Myerson help to a better understanding of how Nash Equilibrium might facilitate the optimization of outcomes in situations that involve numerous decision-makers who have competing interests.

Harsanyi, J. C., & Selten, R. (2018) The interaction of Game Theory and artificial intelligence inside multiagent systems is the subject of emphasis in this particular piece of literature. By delving into computational features, Shoham and Leyton-Brown highlight the role that Nash Equilibrium plays in guiding decision-making in contexts with many agents. Students need to have a solid understanding of the computational components of Nash Equilibrium to successfully apply optimization approaches to real-world issue situations.

METHODOLOGY

This course report on the Nash Equilibrium in Game Theory uses an approach based on a thorough examination of secondary evidence. Existing data obtained for reasons other than the current study is known as secondary data.

Evaluating Nash Equilibrium

Existence

The existence of fixed point theorems for any non-cooperative mixed-strategy finite game has been demonstrated by Nash. In the following phase, I will be leaving this employment. Just so you know, I should emphasize that mixed strategy implies convexity and opens up a broader universe of strategies and beliefs. In other words, it ensures that there will be b.

To begin with, the convexity requirement is fundamental to the majority of fixed point theorems, such as Brower's, Kakutani's, and Schauder's.

Secondly, convexity plays a significant role in mathematical modeling's classifying problem, as well as in linear programming, convex analysis, and the Supported Vector Machine (SVM).

Finally, it is my firm belief that pure mathematics contains the key to understanding convexity. As an example, the Hyperplane Separation Theorem and the Krein-Milman Theorem, which take into account the extreme points in a convex compact set, are both based on locally convex topological space, which is a crucial object in functional analysis. When it comes to LP, I see the extreme points as being analogous to the most fundamentally viable answers.

I still have a lot to learn about convexity, though. More so, because participants in mixed-strategy are allowed to make stochastic strategy choices, it mimics the uncertain reality. By transforming the discrete issue into a continuous one, mixed-strategy makes calculus methods like differential available for NE discovery.

But there aren't many unusual circumstances in which the existence is uncertain. In a noncompact, infinite collection of options, Nash equilibrium is not required to exist. Two people can play a simple game where the winner is the one who names a greater natural number at the same time. A classic example of a price competition scenario in which a NE does not hold is the Bertrand Duopoly, which may be used when the marginal prices of the two enterprises are different.

Uniqueness

The tactics may be predicted with great accuracy if a non-cooperative game has a unique NE since the matching strategy profile will be selected. On the other hand, NE could have some friends if there are any.

There are three Nash equilibria in the following scenario with the background information removed for convenience. Below here, you will find the payout matrix.

Table 3.1: Game with three NE

	Player 2			
		A	**B**	**C**
Player 1	**A**	**0,0**	**20,30**	**5,10**
	B	**37,22**	**0,0**	**5,15**
	C	**8,4**	**13,2**	**9,10**

Identifying the three bold Nash equilibria in this game of pure strategy is not difficult. The values of the payout matrix entries, however, are ones that I purposefully created. We analyze a more fruitful case by creating a mixed-strategy NE from two clear pure-strategy Nash equilibria.

Stability/Invariance

The stability of Nash equilibrium, like that of many other equilibria, deserves a lot of study.

When anything is stable, it stands to reason that even a little disturbance will have little to no effect on the equilibrium, if any effect at all. Parameter, initial, boundary, or "right-hand function" errors should not significantly alter the solution to a partial differential equation (PDE). The stability of equilibrium points, whether it be Lyapunov or global, is another area of interest. The current system of measurements is inherently flawed and imprecise, therefore individuals must rely on approximations while placing a premium on stability to get an accurate picture of the state.

Stability is also critical in real-world applications of Nash equilibria since, as previously said, it is not possible to know with absolute certainty what each player's mixed strategy is; rather, it is necessary to infer it from the distribution of their actions in the game. Keep in mind that the self-enforcement/noncooperative definition of NE makes it such that the Nash equilibrium can only describe stability in terms of unilateral deviations.

RESULTS

Existence of Nash Equilibrium

Theorem 1: The Existence Theorem of Nash In mixed strategies, there exists a Nash equilibrium in every n-player normal-form game when finite strategies apply to all players.

It has been intuitively clear that a mixed-strategy approach is necessary. First, in Chapter 5 of Functional Analysis, I give the abstract of Kakutani's fixed point theorem, omitting the version in Real Analysis.

Theorem 2: Theorem of Fixed Points by Kakutani in Abstract Form Assume that:

(a) A compact convex set K in a locally convex space X that is not empty;

(b) A group G of affine mappings from K to K is an equicontinuous set.

Thus, G and K have a single fixed point.

Sketch of Proof: Contradictory evidence.

First, construct a minimal compact convex set Q invariant under maps of G using either the Hausdorff Maximality Theorem or Zorn's Lemma. Then, apply the equicontinuity to obtain two points x and y whose images cannot be too close under any map T in G.

However, a contradiction is produced since, according to the Krein-Milman Theorem and the condition of the extreme point p, we obtain $x* = y*$ in the closure of the G-orbit of x and y, respectively.

Going back to NE, we outline the set of mixed strategy profiles as $\Delta S = \Delta S1 \times \Delta S2 \times \cdots \times \Delta Sn$, and the collection of best-response correspondences as BR ⋡ BR1 × BR2 Ï……… × BRn. Put simply, BR: $\Delta S \Rightarrow \Delta S$ transforms each element $\sigma \in \Delta S$ into a subset $BR(\sigma) \subset \Delta S$. This truth follows naturally from the previous one:

Fact: A mixed-strategy profile σ ∗ ∈ ∆S is a Nash equilibrium if it is a fixed point of BR, that is $\sigma^* \in BR(\sigma^*)$

Applying the second theorem is particularly challenging because, rather than being a function or map, BR is a multivalued or set-valued function. Using ideas like the analogous class hasn't helped me address this problem. To build endomorphisms, I considered $F_i(\sigma) := (BR_i(\sigma_{-i}), \sigma_{-i})$ This is still not a function, leaving aside the injectivity and surjectivity.

Consequently, dealing with the set-valued function/correspondence is the most challenging step. We offer the following version of Kakutani's fixed point algorithm for this case.

Theorem 3: correspondence and Kakutani's Fixed Point Theorem A fixed point x ∈ C (x) exists in a correspondence C: X ⇒ X when four conditions are met:

- X is a compact, non-empty subset of Rn that is convex.;
- C (x) is non-empty for all x;
- C (x) is convex for all x;
- C has a closed graph.

We next check that the four assumptions of theorem 3 hold:

Function BR: ∆S ⇒ ∆S is applied to ∆S. A non-empty, convex, and compact subset of is X, which is the direct product of n unit cubes in Rmi, if Si is mi in size $R^{\sum_{i=1}^{n} m_i}$.

According to the extreme value theorem, the best-response BRi (σ−i) is not empty since vi (σ−i) is a continuous linear function on the unit cube in Rmi.

Typical Examples

Three well-known games with distinct strategy sets are presented to illustrate the significant findings on NE. Some intriguing games that do not have interval strategies specified include the computational Tragedy of the Commons, the uniqueness-focused Cournot Duopoly, the existence-focused Bertrand Duopoly, and the politically-relevant Electoral Competition.

Prisoner's Dilemma

One of the most famous examples of game theory is the Prisoners' Dilemma, which has several political and economic implications and is sometimes used as a parable. Two players take on the roles of someone under investigation for a major crime, such as an armed robbery, in this static game of full information. Only small-scale theft has been found by the police; to bring the perpetrators of the armed robbery to justice, they require the testimony of at least one of the suspects. The cops decide to play it cleverly by taking separate rooms at the station to interrogate the two suspects. A plea bargain is presented to each suspect with the possibility of a reduced sentence in exchange for a confession—a "fink"—on his accomplice. If the suspect does not want to give the investigators any damning testimony, he might choose to stay silent (M) or say nothing at all.

The reward for each accused party is calculated in the following way:

There is insufficient evidence to sustain any other charges beyond petty theft, so if they both pick their mother, they will both spend two years behind bars. Play1 gets 5 years in jail and Play2 gets 1 year for being the lone cooperator if, for example, Player1 doesn't utter a word while Player2 doesn't do anything. Player 1's finking and Player 2's mumming cause the opposite result. Lastly, if both of them fink, then they will each get a 4-year jail sentence. We choose the reward model that assigns a value of -1 to each year spent in jail since it is logical to believe that more time behind bars is worse. Presented below is its payout matrix:

Table 4.1: Payoff matrix of Prisoner's Dilemma

Prisoner 1	Prisoner 2

		M	F
	M	-2, -2	-5, -1
	F	-1, -5	-4, -4

From what I gather, the following are the main reasons why Prisoner's Dilemma is significant.

- Strict domination can be eloquently shown. With the opponent's strategy set, strategy F is always the superior choice for any player in the game. The first solution concept based on a strongly dominated strategy is IESDS.
- In the same way that the best-response notion may emerge from other solutions, rationalizability can as well. We removed strategy M in our quest to discover rationalizable strategies since it is never the optimum response to any belief in the game.
- The two ideas for solutions that were already presented both depend on reason and the shared understanding of rationality, which are the cornerstones of NE.
- Strategy (M, M) dominates the Pareto optimality, which we may revisit by looking at strategy (F, F) again. This strategy is the only Nash equilibrium.

As an example, I can conclude points 1-3. All tactics are included in the set,

$$\{s|\text{s is a NE}\} \subseteq \{s|\text{s is a } \textit{rationalizable solution}\} \subseteq \{s|\text{s survives IESDS}\}$$

Regarding point 4, the fact that Pareto optimality isn't satisfied suggests that the participants would be better off altering their current environment to implement other forms of enforcement, such as penalizing those who engage in financial misconduct.

If the pain of punishment is z, then for every player who finks, we must deduct z units of payment to clearly understand how this works. Here is a matrix representation of the modified Prisoner's Dilemma.

Table 4.2: Modified Prisoner's Dilemma

	Prisoner 2		
Prisoner 1		M	F
	M	-2, -2	-5, -1-z
	F	-1z, -5	-4z, -4-z

This penalty is sufficient to reverse our anticipated game equilibrium if z is strictly bigger than 1, as M, NE, and Pareto optimality all become the strictly dominant strategies. The taste of authoritarian/uncooperative behavior is plain to see here. Government management is required because institutional design/cooperative contracts can bring participants to the best possible scenario.

Matching Pennies

Each player in a game of matching pennies places two coins on the table at the same time. If both coins land on heads or tails, player 2 wins. If the coins land on different sides, player 1 wins. Player 1 will be given the probability p of playing H and (1−p) of playing T to compute the mixed-strategy Nash equilibrium, whereas player 2 will be given the chance q of playing H and (1 − q) of playing T. Penalty Goal and rock-paper-scissors are two examples of comparable games.

$$v_1\,(H,q) = (-1)\,q + (+1)\,(1-q) = 1-2q$$

Table 4.3: Payoff Matrix of Matching Pennies

Player 1	Player 2		
		H	T
	H	-1, +1	+1, -1
	T	+1, -1	-1, +1

$$v_1\,(T,q) = (+1)\,q + (-1)\,(1-q) = 2q-1$$

$$v_1\,(T,q) = v_1\,(H,q) \implies 1-2q = 2q-1 \implies q = \frac{1}{2}$$

In the same way, $p = \frac{1}{2}.$ Thus, a mixed approach At the Nash equilibrium, players pick H or T at random with $p = \frac{1}{2}$ and $q = \frac{1}{2}.$. An essential NE premise is used in the preceding computation. That is,

Proposition 1 If σ ∗ is a Nash equilibrium, and both si and s' i are in support of σ ∗ i , then

$$v_i\left(s_i, \sigma^*_{-i}\right) = v_i\left(s'_i, \sigma^*_{-i}\right) = v_i\left(\sigma^*_i, \sigma^*_{-i}\right)$$

The notion seems to imply, on the surface, that a player must be agnostic between methods if he is utilizing more than one.

Multiple Equilibria

I propose the following game:

Table 4.4: Payoff Matrix of Matching Penn

Player 1	Player 2		
		C	R
	M	0, 0	3, 5
	D	4, 4	0, 3

Two pure-strategy Nash equilibria may be easily verified: (M, R) and (D, C). When there are two separate pure strategy Nash equilibria in 2 × 2 matrix games such as this one, a third one in mixed strategies will nearly always be present. In this game, the third Nash equilibrium is obtained by a computation analogous to Matching Pennies:

$$\left(\sigma_i^*, \sigma_{-i}^*\right) = \left(\left(\tfrac{1}{6}, \tfrac{5}{6}\right), \left(\tfrac{3}{7}, \tfrac{4}{7}\right)\right).$$

$$BR_1(q) = \begin{cases} p = 1 & \text{if} q \leq \frac{3}{7} \\ p \in [0,1] & \text{if} q = \frac{3}{7} \\ p = 0 & \text{if} q \geq \frac{3}{7}. \end{cases}$$

The same goes for player 2; the optimal answer correspondence is:

$$BR_2(p) = \begin{cases} q = 1 & \text{if} p \leq \frac{1}{6} \\ q \in [0,1] & \text{if} p = \frac{1}{6} \\ q = 0 & \text{if} p \geq \frac{1}{6}. \end{cases}$$

The two best-response correspondences are now drawn according to their appearance in Figure 4.1.

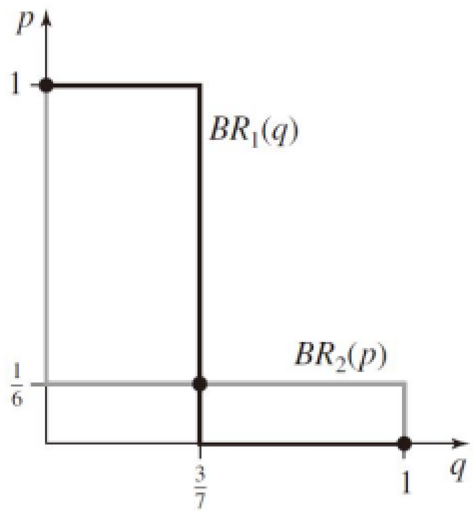

Figure 4.1: Best-response correspondences and Nash equilibria

Figure 4.1 clearly shows all three Nash equilibria.: $(p, q) \in \{(1, 0) \left(\frac{1}{6}, \frac{3}{7}\right) (0, 1)\}.$, It is reasonable to assume that the NE are the unions of the subsets formed by the best-response correspondences of each player in the set of all potential strategy profiles.

CONCLUSION

To sum up, the optimization method class's exploration of Nash equilibrium in game theory has shed light on the mechanics of interactive choice problems and strategic decision-making. An essential idea in game theory, the Nash Equilibrium (so-called because it was first proposed by economist and mathematician John Nash) defines a situation in which, taking into account the strategies used by other players, every player's approach is optimized. We have covered the groundwork of the Nash Equilibrium and seen its practical applications in a variety of contexts, including business, economics, and social interactions. Understanding the depth and breadth of strategic interactions has been enhanced as we have explored the mathematical and visual aspects of Nash Equilibria.

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